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Five-loop ϵ expansion for $\mathrm{U}(n) \times \mathrm{U}(m)$ models: finite-temperature phase transition in light QCD

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Abstract: We consider the $\mathrm{U}(n) \times \mathrm{U}(m)$ symmetric Φ^4 lagrangian to describe the finite-temperature phase transition in QCD in the limit of vanishing quark masses with $n = m = N_f$ flavors and unbroken anomaly at T_c. We compute the Renormalization Group functions to five-loop order in Minimal Subtraction scheme. Such higher order functions allow to describe accurately the three-dimensional fixed-point structure in the plane (n, m), and to reconstruct the line $n^+(m, d)$ which limits the region of second-order phase transitions by an expansion in $\epsilon = 4 - d$. We always find $n^+(m, 3) > m$, thus no three-dimensional stable fixed point exists for $n = m$ and the finite temperature transition in light QCD should be first-order. This result is confirmed by the pseudo-ϵ analysis of massive six-loop three dimensional series.

Keywords: Renormalization Group, Field Theories in Lower Dimensions, QCD.

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1. Introduction

The phase diagram of QCD is characterized by a low temperature hadronic phase with broken chiral symmetry and an high temperature phase with deconfined quarks and gluons, in which chiral symmetry is restored. The nature of the transition between these two phases depends on the QCD parameters, as the number of flavors and quark masses. In the limit of zero quark masses such phase transition is essentially related to the restoring of chiral symmetry (see e.g. the reviews [1]).

The QCD lagrangian with N_f massless quarks is classically invariant under the global flavor symmetry $\mathrm{U}(1)_A \times \mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ [2]. The axial $\mathrm{U}(1)_A$ symmetry may be broken by the anomaly at the quantum level, reducing the relevant symmetry to $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times Z(N_f)_A$ [2]. At $T = 0$ the symmetry is spontaneously broken to $\mathrm{SU}(N_f)_V$ with a nonzero quark condensate. With increasing T, a phase transition characterized by the restoring of the chiral symmetry is expected at T_c. To parameterize this phase transition a complex N_f-by-N_f matrix Φ_{ij} is introduced as an order parameter. The most general renormalizable three-dimensional $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ symmetric lagrangian is [2, 3]

$$\mathcal{L}_{\mathrm{U}(N_f)} = \mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r\, \mathrm{Tr}\, \Phi^\dagger \Phi + \frac{u_0}{4} \left(\mathrm{Tr}\, \Phi^\dagger \Phi\right)^2 + \frac{v_0}{4} \mathrm{Tr} \left(\Phi^\dagger \Phi\right)^2 , \tag{1.1}$$

which describes the QCD symmetry breaking pattern only if $v_0 > 0$ [4].

If the anomaly is broken at T_c, the effective lagrangian is [2, 3]

$$\mathcal{L}_{\mathrm{SU}(N_f)} = \mathcal{L}_{\mathrm{U}(N_f)} + w_0(\det \Phi + \det \Phi^\dagger) . \tag{1.2}$$

The effect of non-vanishing quark masses can be accounted for by adding a linear term $m_{ij}\Phi_{ij}$ in the lagrangian [1, 2, 3], that acts as a magnetic field in a spin model.

The mean-field analysis of the $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ lagrangian eq. (1.1) predicts a second-order phase transition everywhere the stability conditions $v_0 \geq 0$ and $N_f u_0 + v_0 \geq 0$ are satisfied. However, according to Renormalization Group (RG) theory the critical behavior at a continuous phase transition is described by the stable fixed point (FP) of the theory [5]. The absence of a stable FP indicates that the transition cannot be continuous, even though mean-field suggests it. Therefore the transition is expected to be first-order (see e.g. [6]).

The $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ lagrangian was studied at one-loop in $\epsilon = 4 - d$ expansion [2, 7, 8] and at six-loop in the massive zero-momentum renormalization scheme directly in $d = 3$ [4]. No stable FP was found for all values of $N_f \geq 2$, concluding for a first-order phase transition. Anyway both the used approximations have intrinsic limits. The one-loop ϵ expansion, as discussed in ref. [4], provides useful qualitative indications for the description of the RG flow, but it fails in describing quantitatively the right three-dimensional behavior. The major drawback of fixed dimension expansion is that the numerical resummation techniques necessary to extract quantitative informations allow to explore a large, but limited, region in the space of coupling constants (e.g. in ref. [4] the resummation results to be effective only in the region $-2 \leq \overline{u}, \overline{v} \leq 4$, where $\overline{u}, \overline{v}$ are the couplings used in Ref [4]). One cannot exclude a priori that a FP may be outside the accessible region of effectiveness of resummation in $d = 3$. These problems are absent in ϵ expansion since no resummation is needed to find the FP's, being series in ϵ.

For these reasons, we extend the ϵ expansion series to five loops. We consider the $\mathrm{U}(n) \times \mathrm{U}(m)$ generalization of lagrangian (1.1), where Φ is a n-by-m complex matrix. To understand why we decide to study this more general model let us consider the already known one-loop β functions [7]:

$$\beta_u(u,v) = -\epsilon\, u + (nm+4)u^2 + 2(n+m)uv + 3v^2\,, \qquad \beta_v(u,v) = -\epsilon\, v + 6uv + (n+m)v^2\,. \tag{1.3}$$

For $n = m$ a couple of FP's with non-vanishing and negative v exists only for $n < \sqrt{3} + O(\epsilon)$ [2], suggesting a first-order phase transition for those systems, as light QCD, having $v_0 > 0$. Anyway, if one considers the model with $n \neq m$ a more complicated structure of FP's emerges. Two FP's, the gaussian one ($u^* = v^* = 0$) and the $O(2nm)$ one ($v^* = 0$), always exist. For $n \geq n^+(m,d)$ and $n \leq n^-(m,d)$ other two FP's appear which we call U^+ and U^-, whose coordinates at one-loop read

$$u^*_\pm = \frac{A_{mn} \pm (m+n) R_{mn}^{1/2}}{2D_{mn}}\epsilon\,, \qquad v^*_\pm = \frac{B_{mn} \mp 3R_{mn}^{1/2}}{D_{mn}}\epsilon\,, \tag{1.4}$$

with

$$\begin{aligned} B_{mn} &= nm^2 - 5n + mn^2 - 5m\,, \quad & A_{mn} &= 36 - m^2 - 2mn - n^2\,, \\ R_{mn} &= 24 + m^2 - 10mn + n^2\,, \quad & D_{mn} &= 108 - 8m^2 - 16mn + m^3 n - 8n^2 + 2m^2n^2 + mn^3\,. \end{aligned} \tag{1.5}$$

The v^* coordinates of $U^\pm$ for $n > n^+$ are positive, as it should be to provide the right symmetry breaking of QCD. Requiring $R_{mn} > 0$, we have $n^\pm = 5m \pm 2\sqrt{6}\sqrt{m^2-1} + O(\epsilon)$.[1]

[1]Note that it is possible to study the generalized $\mathrm{U}(n) \times \mathrm{U}(m)$ model at fixed m in a $1/n$ expansion, since it has a FP for $n = \infty$, contrarily to the $\mathrm{U}(n) \times \mathrm{U}(n)$.

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The stability properties of these FP's at fixed m (for the physically relevant case $m \geq 1$) are characterized by the following four different regimes (note the analogy with the $O(n) \times O(m)$ model [9]):

1. For $n > n^+(m,d)$, there are four FP's, and U^+ is the only stable. Both $U^\pm$ have $v^* > 0$.
2. For $n^-(m,d) < n < n^+(m,d)$, only the gaussian and the Heisenberg O($2nm$)-symmetric FP's are present, and none of them is stable. Thus the transition is expected to be first-order.
3. For $n_H(m,d) < n < n^-(m,d)$, there are again four FP's, and U^+ is the stable one. However at small ϵ for $m < 5$, it has $v^* < 0$. For $v_0 > 0$ a first-order transition is expected. For this reason we will never consider the value of $n^-(m,d)$.
4. For $n < n_H(m,d)$,[2] there are again four FP's, and the Heisenberg O($2mn$)-symmetric one is stable.

Now it is possible (and one has to check!) that the actual value of $n^+(m,3)$ is lower than m, providing a stable FP and consequently a new universality class for $\mathrm{U}(N_f)\times\mathrm{U}(N_f)$ symmetric models. To give a definitive answer to this question, high order calculations are required, since low order ones lead to erroneous conclusions, as we shall see. However we anticipate that we do not find any stable FP, supporting the results of ref. [4].

The paper is organized as follows. The $\mathrm{U}(n) \times \mathrm{U}(m)$ model is analyzed at five-loop in ϵ expansion in section 2. In section 3, the model is reanalyzed with pseudo-ϵ expansion at six-loop order in massive zero-momentum renormalization scheme. Section 4 summarizes our main results. In the appendix we briefly discuss the effect of the anomaly.

2. Five-loop ϵ-expansion of $\mathrm{U}(n) \times \mathrm{U}(m)$ model

We extend the one-loop ϵ expansion of refs. [7] for the RG functions of the $\mathrm{U}(n) \times \mathrm{U}(m)$ symmetric theory to five-loop. For this purpose, we consider the minimal subtraction ($\overline{\mathrm{MS}}$) renormalization scheme [5] for the massless theory. We compute the divergent part of the irreducible two-point functions of the field Φ, of the two-point correlation functions with insertions of the quadratic operators Φ^2, and of the two independent four-point correlation functions. The diagrams contributing to this calculation are 162 for the four-point functions and 26 for the two-point one. We handle them with a symbolic manipulation program, which generates the diagrams and computes the symmetry and group factors of each of them. We use the results of ref. [13], where the primitive divergent parts of all integrals appearing in our computation are reported. We determine the renormalization constant Z_Φ associated with the fields Φ, the renormalization constant Z_t of the quadratic operator Φ^2,

[2] The value of $n_H(m)$ may be inferred from refs. [10, 11], where it was shown, on the basis that at a global $O(N)$ FP all the spin four operators have the same scaling dimension, that the $O(N)$ FP is stable only if $N < N_c$, with $N_c \sim 2.9$ [6, 11, 12]. Thus the $O(2nm)$ FP is stable for $n \lesssim 1.45/m$ and it is not expected to play any role at the QCD phase transition.

and the renormalized quartic couplings u, v. The functions β_u, β_v, η_ϕ and η_t are determined using the relations

$$\beta_u(u,v) = \mu \frac{\partial u}{\partial \mu}\bigg|_{u_0,v_0}, \qquad \beta_v(u,v) = \mu \frac{\partial v}{\partial \mu}\bigg|_{u_0,v_0}, \tag{2.1}$$

$$\eta_\phi(u,v) = \frac{\partial \log Z_\Phi}{\partial \log \mu}\bigg|_{u_0,v_0}, \qquad \eta_t(u,v) = \frac{\partial \log Z_t}{\partial \log \mu}\bigg|_{u_0,v_0}. \tag{2.2}$$

The zeroes (u^*, v^*) of the β functions provide the FP's of the theory. In the framework of the ϵ expansion, they are obtained as perturbative expansions in ϵ and then are inserted in the RG functions to determine the ϵ expansion of the critical exponents:

$$\eta = \eta_\phi(u^*, v^*), \qquad \nu = (2 - \eta_\phi(u^*, v^*) - \eta_t(u^*, v^*))^{-1}. \tag{2.3}$$

2.1 RG functions

The five-loop expansions of the β functions are given by

$$\begin{aligned}
\beta_u = &-\epsilon\, u + (nm+4)u^2 + 2(n+m)uv + 3v^2 - \frac{3}{2}(7+3mn)u^3 - 11(m+n)u^2 v - \\
&-\frac{41+5mn}{2}uv^2 - 3(m+n)v^3 + \\
&+\left(\frac{740+461mn+33m^2n^2}{16} + \zeta(3)(33+15mn)\right)u^4 + \\
&+\left(\frac{659n+79m^2n+659m+79mn^2}{8} + 48\zeta(3)(m+n)\right)u^3 v + \\
&+\left(\frac{2619+1210mn+230n^2+230m^2+3n^2m^2}{16} + 18\zeta(3)(7+mn)\right)u^2v^2 + \\
&+\left(\frac{15}{4}(20n+m^2n+20m+n^2m) + 36\zeta(3)(m+n)\right)uv^3 + \\
&+\left(\frac{425+20m^2+153mn+20n^2}{16} + 6\zeta(3)(4+mn)\right)v^4 + \beta_5^u + \beta_6^u,
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\beta_v(u,v) = &-\epsilon\, v + 6uv + (n+m)v^2 - \frac{41+5mn}{2}u^2 v - 11(m+n)uv^2 - 3\frac{5+mn}{2}v^3 + \\
&+v\Bigg[\left(\frac{821+184mn-13m^2n^2}{8} + 12\zeta(3)(7+mn)\right)u^3 + \\
&\quad+\left(\frac{1591\,n - 35\,m^2\,n + 1591m - 35mn^2}{16} + 72\zeta(3)(m+n)\right)u^2 v + \\
&\quad+\left(\frac{211+9\,m^2+73mn+9n^2}{2} + 24\zeta(3)(4+mn)\right)uv^2 + \\
&\quad+\left(\frac{295n+13m^2n+295m+13mn^2}{16} + 9\zeta(3)(m+n)\right)v^3\Bigg] + \beta_5^v + \beta_6^v.
\end{aligned} \tag{2.5}$$

The coefficients $\beta_5^u, \beta_6^v, \beta_5^u, \beta_6^u$ are very long and not really illuminating. We do not report them here, but they are available on request to the authors. The same is true for the RG functions η_ϕ and η_t to five-loop, that we calculated but never used, since we did not find evidence for a FP in the space of parameters of interest.

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We have checked that for $v = 0$ the series reduce to the existing $O(\epsilon^5)$ ones for the O($2nm$)-symmetric theory [14]. For $m = 1$ and any n (or viceversa, given the symmetry under the exchange $n \leftrightarrow m$) the theory is equivalent to an $O(2n)$ in the coupling $u + v$, so the series satisfy the relation $\beta_u(z+y, z-y; n, m = 1)+\beta_v(z+y, z-y; n, m = 1) = \beta_{O(2n)}(z)$, where $\beta_{O(2n)}(z)$ is the β-function of the O($2n$) model [14].

2.2 Estimates of $n^+(m, 3)$

From the above reported series, the ϵ expansion of $n^{\pm}(m)$ may be calculated to $O(\epsilon^5)$. $n^{\pm}(m)$ may be expanded as

$$n^{\pm}(m) = n_0^{\pm}(m) + n_1^{\pm}(m)\,\epsilon + n_2^{\pm}(m)\,\epsilon^2 + n_3^{\pm}(m)\,\epsilon^3 + n_4^{\pm}(m)\,\epsilon^4 + O(\epsilon^5)\,, \tag{2.6}$$

and the coefficients $n_i^{\pm}(m)$ are obtained by requiring

$$\beta_u(u^*, v^*; n^{\pm}) = 0\,, \qquad \beta_v(u^*, v^*; n^{\pm}) = 0\,, \qquad \text{and} \qquad \det\left|\frac{\partial(\beta_u, \beta_v)}{\partial(u, v)}\right|(u^*, v^*; n^{\pm}) = 0\,. \tag{2.7}$$

For generic values of m, the expression of $n^{\pm}(m, 4-\epsilon)$ is too cumbersome in order to be reported here. We only report the numerical expansion of n^+ at fixed $m = 2,\ 3, 4$, i.e.

$$\begin{aligned}
n^+(2, 4-\epsilon) &= 18.4853 - 19.8995\epsilon + 2.9260\epsilon^2 + 4.6195\epsilon^3 - 0.7182\epsilon^4 + O(\epsilon^5)\,,\\
n^+(3, 4-\epsilon) &= 28.8564 - 30.0833\epsilon + 6.5566\epsilon^2 + 3.4056\epsilon^3 - 0.7958\epsilon^4 + O(\epsilon^5)\,,\\
n^+(4, 4-\epsilon) &= 38.9737 - 40.2386\epsilon + 9.6089\epsilon^2 + 3.0505\epsilon^3 - 0.6156\epsilon^4 + O(\epsilon^5)\,.
\end{aligned} \tag{2.8}$$

In order to give an estimate of $n^+(m, 3)$ such series should be evaluated at $\epsilon = 1$. A linear extrapolation of the two-loop contribution leads to the wrong conclusion that $n^+(m, 3) < m$, i.e. that the transition is continuous. This is the anticipated statement that high-loop computations are needed to have a conclusive result. Anyway, a direct sum of the five-loop series is not effective, since they are expected to be divergent. The high irregular behavior with the number of the loops makes also a Borel-like resummation non effective as well (in fact Padé-Borel resummation leads to unstable results). Thus we try to extract from eqs. (2.8) better behaved series by means of algebraic manipulations.

This may be done considering (as in ref. [15])

$$\begin{aligned}
1/n^+(2, 4-\epsilon) &= 0.0541 + 0.0582\epsilon + 0.0541\epsilon^2 + 0.0355\epsilon^3 + 0.0172\epsilon^4 + O(\epsilon^5)\,,\\
1/n^+(3, 4-\epsilon) &= 0.0347 + 0.0361\epsilon + 0.0298\epsilon^2 + 0.0188\epsilon^3 + 0.0095\epsilon^4 + O(\epsilon^5)\,,\\
1/n^+(4, 4-\epsilon) &= 0.0257 + 0.0265\epsilon + 0.0210\epsilon^2 + 0.0132\epsilon^3 + 0.0067\epsilon^4 + O(\epsilon^5)\,,
\end{aligned} \tag{2.9}$$

whose coefficients decrease rapidly. Setting $\epsilon = 1$ we obtain the results reported in table 1.

Another method, firstly employed for $O(m) \times O(n)$ models in ref. [16], use the knowledge of $n^+(m, 2)$ to constrain the analysis at $\epsilon = 2$, under the (strong) assumption that $n^+(m, d)$ is sufficiently smooth in d at fixed m. $n^+(m, 2)$ may be conjectured further assuming that the two-dimensional LGW stable FP is equivalent to that of the NLσ model for all $n \geq 1$ except $n = 1$ [5]. Since the NLσ model is asymptotically free, we conclude that $n^+(m, 2) = 1$. The knowledge of $n^+(m, 2)$ may be exploited in order to obtain

m	$1/n^+$			a			final
	3-loop	4-loop	5-loop	3-loop	4-loop	5-loop	
2	6.01	4.95	4.56	4.98	4.55	4.44	4.5(5)
3	9.94	8.38	7.76	8.07	7.53	7.40	7.6(8)
4	13.7	11.6	10.7	11.0	10.3	10.2	10.5(1.1)

Table 1: Estimates of n^+ for several m with varying the number of loops.

some informations on $n^+(m,3)$, rewriting the perturbative series for $n^+(m,4-\epsilon)$ in the following form

$$n^+(m,4-\epsilon) = 1 + (2-\epsilon)\, a(m,\epsilon)\,, \tag{2.10}$$

where

$$\begin{aligned} a(2,\epsilon) &= 8.743 - 5.579\epsilon - 1.326\epsilon^2 + 1.646\epsilon^3 + 0.464\epsilon^4 + O(\epsilon^5)\,, \\ a(3,\epsilon) &= 13.928 - 8.078\epsilon - 0.760\epsilon^2 + 1.323\epsilon^3 + 0.263\epsilon^4 + O(\epsilon^5)\,, \\ a(4,\epsilon) &= 18.987 - 10.626\epsilon - 0.508\epsilon^2 + 1.271\epsilon^3 + 0.328\epsilon^4 + O(\epsilon^5)\,, \end{aligned} \tag{2.11}$$

whose terms are better behaved than the original series, but not decreasing. We consider the series $a(2,\epsilon)^{-1}$ obtaining a more "convergent" expression, which may be estimated simply by setting $\epsilon = 1$. The results of this constrained analysis are reported in table 1. Note that, although the several not completely justified assumptions we made, the final obtained series are highly stable with changing the number of the loops. Obviously this is not an evidence favoring the validity of the assumptions, but it is a very convincing argument to ensure the goodness of our estimates.

As final estimates we quote an average of the five-loop results (that are quite close) and as error bar the maximum difference with the fourth order ones. For all the considered value of m we have $n^+(m,3) > m$, thus the transition for $\mathrm{U}(n)\times\mathrm{U}(n)$ models is expected to be first-order. We also check that $n^+(m,3) > m$ for higher values of m.

Since the new couple of FP's does not exist for finite temperature transition of light QCD, we do not report their expansion in terms of ϵ and the exponents characterizing the critical behavior for $n > n^+$. Anyway they may be obtained from the series we reported and from those that are available on request.

3. Pseudo-ϵ expansion

In this section we analyze the six-loop zero-momentum massive three-dimensional series with the so-called pseudo-ϵ expansion method [17], since it provided the best results for the marginal spin dimensionality in spin models (see e.g. refs. [15, 6] and references therein). The β functions for generic n and m were calculated at six-loop in ref. [4] (but they were not reported there).

The idea behind the pseudo-ϵ expansion is very simple [17]: one has only to multiply the linear terms of the two β functions by a parameter τ, find the common zeros of the β's as series in τ and analyze the results as in the ϵ expansion. The critical exponents are

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	$N=0$	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$
$M=0$	18.485	5.219	4.769	4.596	4.542	4.527
$M=1$	10.762	$4.753_{[29.5]}$	$4.487_{[2.59]}$	$4.518_{[3.23]}$	$4.522_{[3.72]}$	
$M=2$	8.190	$4.579_{[7.43]}$	$4.518_{[3.21]}$	4.523		
$M=3$	6.921	$4.533_{[4.99]}$	$4.522_{[3.65]}$			
$M=4$	6.181	$4.524_{[4.30]}$				
$M=5$	5.708					

Table 2: Padé table for $n^+(2,3)$ in pseudo-ϵ expansion. The location of the positive real pole closest to the origin is reported in brackets.

obtained as series in τ inserting the FP's expansions in the appropriate RG functions. Note that, differently from the ϵ expansion, only the value at $\tau = 1$ makes sense. The reason for which it works well is twofold: first in the three dimensional approach at least one order more in the loop expansion is known, second, and more important, the RG functions are better behaved in the massive approach [5, 17].

Following the recipe explained in the previous section for the $\overline{\mathrm{MS}}$ scheme, we obtain

$$\begin{aligned}
n^+(2,3) &= 18.4853 - 13.2663\tau - 0.4499\tau^2 - 0.1735\tau^3 - 0.0537\tau^4 - 0.0144\tau^5 \,, \\
n^+(3,3) &= 28.8564 - 20.0555\tau - 0.3092\tau^2 - 0.2609\tau^3 - 0.1444\tau^4 - 0.0968\tau^5 \,, \\
n^+(4,3) &= 38.9737 - 26.8257\tau - 0.2690\tau^2 - 0.3582\tau^3 - 0.2199\tau^4 - 0.1526\tau^5 \,. \quad (3.1)
\end{aligned}$$

At least up to the known order, such expansions do not behave as asymptotic with factorial growth of coefficients and alternating signs. So one may apply a simple Padé resummation [15]. The results of the $[N/M]$ Padé approximants are displayed in table 2 for $m = 2$. Several approximants have poles on the positive real axis. Anyway all these poles are "far" from $\tau = 1$, where the series must be evaluated. Thus one may expect the presence of a pole not to influence the result at $\tau = 1$. Anyway for the cases $m = 3, 4$ some Padé have poles at $\tau < 2$, that must be discarded in the average procedure. We choose as final estimate the average the six-loop order Padé without poles at $\tau < 2$ (excluding those with $N = 0$, giving unreliable results), and as error bar we take the maximum deviation from the average of four- and five-loop Padé (as in ref. [15]). Within this procedure we have $n^+(2,3) = 4.52(7)$, $n^+(3,3) = 7.98(25)$, $n^+(4,3) = 11.1(4)$.

The final results are in very good agreement with those of the previous section from a completely different approach. This is a clear evidence that the estimates we made are robust.

4. Conclusions

In this paper we investigated the possibility of a second-order phase transition in QCD in the limit of vanishing masses. When the $\mathrm{U}(1)_A$ symmetry is restored at T_c, the finite temperature chiral transition, if continuous, may be described by the lagrangian $\mathcal{L}_{\mathrm{U}(N_f)}$ eq. (1.1). We pointed out that the correct extrapolation at $\epsilon = 1$ is obtainable considering

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its U(n) $\times$ U(m) symmetric extension at fixed m. In fact the last model has a stable FP for $n > n^+(m,d)$ that is not accessible from the theory with $n = m$. The presence of a stable FP for light QCD with $N_f = m$ flavors requires $n^+(m,3) < m$. After showing that low order calculations are not conclusive, we performed a five-loop expansion that allowed to conclude that no continuous transition is possible for three dimensional models with U(n) $\times$ U(n) symmetry for $n \geq 2$. We corroborated this result with a direct three-dimensional analysis, namely with the pseudo-ϵ expansion at six-loop order.

In ref. [4] six-loop massive zero momentum series were analyzed directly in three dimensions allowing to exclude that a FP, without any counterpart in ϵ expansion exists (as it was claimed to happen for $O(n) \times O(2)$ models for low values of n both for $v > 0$ [18] and $v < 0$ [19]). We believe that this work, together with ref. [4], put on a robust basis the prediction that the transition in U(n) $\times$ U(n) models is first-order.

In the appendix, following ref. [4], we point out that the anomaly may lead to a continuous transition only for $N_f = 2$ and large values of $|w_0|$, instead for $N_f \geq 3$ it does not softens the first-order transition of the U(N_f) $\times$ U(N_f) model, for any value of w_0.

Finally it is worth mentioning that the U(n) $\times$ U(m) models could be relevant in the description of some quantum phase transitions, as it happens for their $O(n) \times O(m)$ counterparts for Mott insulators [20]. Being $n^+(m,3) > m$ for all $m \geq 2$, we predict a first-order phase transition for all those systems with $m \leq 3$, which are interesting for the condensed matter point of view.

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A. The effect of the anomaly

If the U$(1)_A$ symmetry is broken at T_c by the anomaly, one has to consider the lagrangian (1.2). Since the effect of the anomaly is always (apart from the case $N_f = 4$) well described by general arguments, this appendix is very similar to part of ref. [4]. However, we report such arguments here in order to make this paper self-consistent.

In the large-N_c limit (where N_c is the number of colors), the effect of the anomaly tends to be suppressed, and the lagrangian eq. (1.1) is recovered in the limit $N_c \to \infty$. The effective U$(1)_A$-symmetry breaking at finite temperature in real QCD has been investigated on the lattice. The U$(1)_A$ symmetry appears not to be restored at T_c, but the effective breaking of the axial U$(1)_A$ symmetry appears substantially reduced especially above T_c (see, e.g., refs. [21]). However, the lagrangian (1.1) still describe a large part of the phase diagram of the model with broken anomaly as we show closely following ref. [4].

For $N_f = 2$ the symmetry breaking pattern is equivalent to $O(4) \to O(3)$ [2]. If the transition is continuous it is in the three-dimensional $O(4)$ universality class [2, 3, 22, 23], which has been accurately studied in the literature [6, 24]. Actually a continuous transition is expected only for large enough value of $|w_0|$ (see ref. [4], in particular appendix A, for the phase diagram of this model). In particular the multicritical point is $\mathrm{U}(2) \times \mathrm{U}(2)$ symmetric. The phase diagram realized in light QCD may be understood only from the QCD lagrangian and not from universality arguments. Lattice simulations in two flavors QCD favor a continuous transition consistent with the $O(4)$ universality class [25].

For $N_f = 3$ the determinant is cubic in Φ, making the lagrangian not bounded. So the transition is expected to be first-order for all w_0. Lattice simulations of QCD confirm this expectation [26].

For $N_f = 4$ the determinant is quartic in Φ, leading to the three couplings effective Φ^4 lagrangian

$$\mathcal{L}_{\mathrm{SU}(4)} = \mathcal{L}_{\mathrm{U}(4)} + w_0 \epsilon_{ijkl} \epsilon_{abcd} \Phi_{ia} \Phi_{jb} \Phi_{kc} \Phi_{ld} \,, \tag{A.1}$$

where ϵ_{ijkl} is the completely antisymmetric tensor ($\epsilon_{1234} = 1$). Such lagrangian is not generalizable to an n-by-m matrix with n or m different from 4 and so we limit to consider the case $n = m = 4$. The one-loop β functions we obtain are

$$\begin{aligned}
\beta_u(u,v,w) &= -\epsilon u + 20u^2 + 16uv + 3v^2 + 8w^2 \,, \\
\beta_v(u,v,w) &= -\epsilon v + 8v^2 + 6uv - 8w^2 \,, \\
\beta_w(u,v,w) &= -\epsilon w + 6uw - 6vw \,.
\end{aligned} \tag{A.2}$$

Such series, as in six-loop three-dimensional case [4], have no common zeros with non-vanishing coordinates. Anyway, differently from three dimensions, the β_w vanishes in a region of parameters, different from $w = 0$ (namely the surface $\epsilon - 6(u - v) = 0$). Higher loop corrections may not change the number of FP's, since they are expansions in ϵ.

For $N_f \geq 5$ the added anomaly term is irrelevant since it generates polynomials of degrees higher than four. Therefore for $N_f \geq 5$ the lagrangians $\mathcal{L}_{\mathrm{SU}(N_f)}$ and $\mathcal{L}_{\mathrm{U}(N_f)}$ are equivalent at criticality.

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